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Theory of Measurement in Quantum Mechanics and Macroscopic Description

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The general formulation of quantum mechanics based on the concept of effect and operation is briefly recalled and the problem of the consistency of the shift of the borderline between object and observer is discussed. The formalism for the continuous observations based on the concept of an operation-valued stochastic process is also reviewed and it is shown on a simple model how it can be applied to an objective macroscopic description of a system of many particles. The relevance to the problem of the interpretation of quantum mechanics is discussed, but difficulties related to the conservation laws and to a relativistic extension are also pointed out.

1. INTRODUCTION

In this paper I wish to review briefly the general formulation of quantum mechanics based on the concepts of *effect* and *operation* (Ludwig, 1982; Kraus, 1983; Davies, 1976; Holevo, 1982*a*) to stress that it solves in precise mathematical terms the problem of the consistency of the shift of the demarcation line between objects and apparatus as formulated by von Neumann (1955). The important fact is that such solution is obtained without the unnatural and unsatisfactory assumptions on the working of the apparatus that are usually adopted in a different context. I shall also recall how the collapse of the wave function can be naturally understood in the same perspective.

Then I shall discuss the formalism of the *continuous observations* in quantum mechanics (Barchielli *et al.*, 1982, 1983, 1984; Barchielli and Lupieri, 1985, 1986; Davies, 1969, 1970, 1971; Holevo, 1988, 1989; Prosperi,

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1987) [for applications of the formalism see Barchielli (1983, 1987, 1988, 1990), Srinivas and Davies (1981, 1982), and Holevo (1982*b*)] and show how this can be applied to examples of *macroscopic variables.* I shall notice that, if, with a change of language such variables are understood as *be-ables* in the sense of Bell (that is, as having a well-defined value at any time), rather than as specifying quantities continuously taken under observation, the problem of an objective macroscopic level of description of the physical world can be solved positively even in the context of quantum mechanics, by an explicit realization of the Ludwig program. Unfortunately, this solution cannot be considered satisfactory, since, at least for the simplest models we have been able to produce, it falls in conflict with basic conservation laws.

In spite of the difficulties, I shall try to pursue the program explicitly to a certain extent on a simple model. Indeed, it develops completely in the framework of ordinary quantum mechanics and requires only a generalization of the concept of observable. I think it should have at least a pedagogic value. In the model the be-able shall be simply the density of particles, starting from which, however, the entire thermodynamics can be constructed in principle. A more sophisticated attempt is presented by Lanz elsewhere in these Proceedings.

The discussion concerning the last subject, reported in Section 6, has not been published before; the rest of the paper is intended simply as an updating and an introduction to the work of Lanz and Barchielli.

2. GENERAL FORMALISM

In the general formulation of quantum mechanics mentioned above a set of p compatible quantities, abstractly denoted by $A = (A_1, A_2, \ldots, A_p)$, is associated to an *effect-valued measure* (e.v.m.) $\hat{F}_A(T)$ on $\mathbf{R}^p$, and an apparatus S_A for observing A to a similar *operation-valued measure* (o.v.m.) $\mathscr{F}_{S_A}(T)$ (in this connection the term *instrument* and the symbol $\mathscr{I}(T)$ are also frequently used).

I recall that by the terms *effect* and *operation* we mean a bounded self-adjoint operator in the Hilbert space $\mathbf{H}$ with the property

$$0 \leq \hat{F} \leq \hat{I} \tag{2.1}$$

and a linear mapping of the space of the *trace class operators* $\mathbf{T(H)}$ into itself which is *completely positive* and *trace decreasing,*

$$\mathrm{Tr}(\mathscr{F}\hat{X}) \leq \mathrm{Tr}\,\hat{X} \tag{2.2}$$

respectively. Similarly, by an e.v.m. and an o.v.m. we mean a mapping from the Borel sets $B(\mathbf{R}^p)$ on $\mathbf{R}^p$ into the family of the effects or of the operations respectively, such that

$$\hat{F}\left(\bigcup_{j=1}^{\infty} T_j\right) = \sum_{j=1}^{\infty} \hat{F}(T_j) \quad \text{and} \quad \mathscr{F}\left(\bigcup_{j=1}^{\infty} T_j\right) = \sum_{j=1}^{\infty} \mathscr{F}(T_j) \tag{2.3}$$

for $T_i \cap T_j = \varnothing$ if $i \neq j$.

The specific e.v.m. $\hat{F}_A(T)$ and o.v.m. $\mathscr{F}_{S_A}(T)$ associated to A and S_A must be supposed normalized,

$$\hat{F}_A(\mathbf{R}^p) = I, \qquad \operatorname{Tr} \mathscr{F}_{S_A}(\mathbf{R}^p)\hat{X} = \operatorname{Tr} \hat{X} \tag{2.4}$$

and related by the equation

$$\hat{F}_A(T) = \mathscr{F}'_{S_A}(T)\hat{I} \tag{2.5}$$

where $\mathscr{F}'$ denotes the adjoint of $\mathscr{F}$ (remember that the dual space of $\mathbf{T}$ is the space $\mathbf{B}$ of the bounded operators in $\mathbf{H}$), i.e.,

$$\operatorname{Tr}(\hat{F}_A(T)\hat{X}) = \operatorname{Tr}(\mathscr{F}_{S_A}(T)\hat{X}) \tag{2.6}$$

Obviously there are *many* o.v.m.'s corresponding to *one* e.v.m. by (2.5) or (2.6). This agrees with the fact that we may conceive many different kinds of apparatus S_A for observing the same set of quantities A.

In the Heisenberg picture we set

$$\hat{F}_A(T, t) = e^{i\hat{H}t}\hat{F}_A(T)e^{-i\hat{H}t} \tag{2.7a}$$

$$\mathscr{F}_{S_A}(T, t)\hat{X} = e^{i\hat{H}t}[\mathscr{F}_{S_A}(T)(e^{-i\hat{H}t}\hat{X}e^{i\hat{H}t})]e^{-i\hat{H}t} \tag{2.7b}$$

Then the probability of observing a set of values $A \in T$ by the apparatus S_A at a time t_1 is assumed to be given by

$$P(A \in T, t_1 | W) = \operatorname{Tr}(\hat{F}_A(T, t_1)\hat{W}) = \operatorname{Tr}[\mathscr{F}_{S_A}(T, t_1)\hat{W}] \tag{2.8}$$

$\hat{W}$ being the *statistical* (or *density*) *operator* representing the state of the system. Furthermore, if the result $A \in T$ has been actually found, the state of the system is modified in the following way:

$$\hat{W} \to \mathscr{F}_{S_A}(T, t_1)\hat{W}/\operatorname{Tr}[\mathscr{F}_{S_A}(T, t_1)\hat{W}] \tag{2.9}$$

Obviously ordinary textbook quantum mechanics is recovered by requiring the e.v.m. $\hat{F}_A(T)$ be a *projection valued measure.*

Note also that from equation (2.8) we obtain for the expectation values of the various A_s

$$\langle A_s \rangle = \operatorname{Tr}\{\hat{O}_s(t)\hat{W}\} \tag{2.10}$$

having set

$$\hat{O}_s(t) = e^{i\hat{H}t}\hat{O}_s e^{-i\hat{H}t} \quad \text{and} \quad \hat{O}_s = \int_{\mathbf{R}^p} d\hat{F}(x)x_s \tag{2.11}$$

So even in the present formulation a set of symmetric (even if not necessarily self-adjoint) operators is associated to a set of compatible observables. There are, however, two important differences from the ordinary formulation: (a) in general the operators $\hat{O}_1, \ldots, \hat{O}_p$ do not commute with each other [in general $\hat{F}_A(T)\hat{F}_A(S) \neq \hat{F}_A(S)\hat{F}_A(T)$]. (b) Since now the decomposition (2.11) is no longer unique, there are many e.v.m.'s, and so many different sets of compatible observables $A \equiv (A_1, \ldots, A_p)$ associated to the same set of operators. Obviously all such sets correspond to a single set of classical quantities.

In some sense we could think of the above $A_1, \ldots, A_p$ as corresponding to a kind of simultaneous coarse-grained observation of the p commuting or noncommuiting quantities associated to $\hat{O}_1, \ldots, \hat{O}_p$ in the ordinary formulation.

3. THE PROCESS OF MEASUREMENT

Let us now introduce explicitly in the treatment the apparatus by which a certain observation on the system is performed.

We mean by apparatus a second system which interacts for a certain time with the object and which is affected by this interaction in an appreciable way.

Denoting the object by I and the apparatus by II, we write the Hamiltonian of the compound system as

$$\hat{H} = \hat{H}_{\rm I} + \hat{H}_{\rm II} + \hat{H}_{\rm int} = \hat{H}_0 + \hat{H}_{\rm int} \tag{3.1}$$

Then we can assimilate the interaction between I and II to a scattering process and assume that the limit

$$\lim_{t'' \to +\infty,\, t' \to -\infty} e^{i\hat{H}_0 t''} e^{-i\hat{H}(t''-t')} e^{-i\hat{H}_0 t'} = \hat{U} \tag{3.2}$$

exists in the strong or in the weak sense.

We denote by $A_{\rm II}$ the position of an index or any other quantity by which we measure the modifications that occur in II and by $\hat{F}^A_{\rm II}(T)$ the e.v.m. corresponding to $A_{\rm II}$ according to the rules of the preceding section. We also assume that II is in the state $W_{\rm II}$ and I in the state $W_{\rm I}$ before the interaction, that at the time t_0 the interaction is finished, and that then the observation of $A_{\rm II}$ is performed. We obtain

$$P(A_{\rm II} \in T, t_0 | W_{\rm I} W_{\rm II}) = {\rm Tr}[\hat{F}^A_{\rm II}(T, t_0)\hat{U}\hat{W}_{\rm I}\hat{W}_{\rm II}\hat{U}^*] \tag{3.3}$$

where we have used the interaction picture, i.e., we have written

$$\hat{F}^A_{\rm II}(T, t) = \exp(i\hat{H}_{\rm II} t)\, \hat{F}^A_{\rm II}(T) \exp(-i\hat{H}_{\rm II} t)$$

Taking into account the relation

$$e^{-i\hat{H}_0 t}\hat{U} = \hat{U}e^{-i\hat{H}_0 t} \tag{3.4}$$

which follows from equation (3.2), we can rewrite the right-hand side of equation (3.3) as

$$P(A_{\rm II}\in T|W_{\rm I}W_{\rm II}) = {\rm Tr}^{\rm I}\{\hat{F}_{\rm I}^A(T, t_0)\hat{W}_{\rm I}\} = P(A_{\rm I}\in T, t_0|W_{\rm I}) \tag{3.5}$$

Here

$$\hat{F}_{\rm I}^A(T, t_0) = e^{i\hat{H}_{\rm I}t_0}\,{\rm Tr}^{\rm II}\{\hat{W}_{\rm II}^{1/2}e^{i\hat{H}_{\rm II}t_0}\hat{U}^*\hat{F}_{\rm II}^A(T)\hat{U}e^{-i\hat{H}_{\rm II}t_0}\hat{W}_{\rm II}^{1/2}\}e^{-i\hat{H}_{\rm I}t_0} \tag{3.6}$$

is obviously an e.v.m. and Tr, $\rm Tr^I$, and $\rm Tr^{II}$ denote trace operations performed on $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$, $\mathbf{H}_1$, and $\mathbf{H}_2$ which are the Hilbert spaces of the compound system, of system I alone, and of the system II alone, respectively.

Equation (3.5) shows that the observation of the quantity $A_{\rm II}$ on II at time t_0 after the interaction can be equivalently described as the observation of the quantity $A_{\rm I}$ on I corresponding to the e.v.m. $\hat{F}_{\rm I}^A(T)$ and essentially it proves the consistency of the postulates with a reasonable general characterization of the apparatus.

Note that, even if $\hat{F}_{\rm II}^A(T)$ is supposed to be a projection-valued measure, in general $\hat{F}_{\rm I}^A(T)$ turns out to be an effect-valued measure unless very special and unrealistic assumptions on $\hat{U}$ are done.

Suppose we perform a second observation on I at a subsequent time t. If we do not introduce the new apparatus explicitly in the treatment, we can write

$$P(B_{\rm I}\in S, t; A_{\rm II}\in T, t_0|W_{\rm I}W_{\rm II}) = {\rm Tr}\{\hat{F}_{\rm I}^B(S, t)\hat{F}_{\rm II}^A(T, t_0)\hat{U}\hat{W}_{\rm I}\hat{W}_{\rm II}\hat{U}^\dagger\} \tag{3.7}$$

for the joint probability of observing $A_{\rm II}\in T$ at t_0 and $B_{\rm I}\in S$ at t. Here $B_{\rm I}$ denotes the new observable and $\hat{F}_{\rm I}^B(S)$ the corresponding e.v.m.; we have taken into account the fact that $\hat{F}_{\rm I}^B(S, t) = \exp(i\hat{H}_{\rm I}t)\,\hat{F}_{\rm I}^B(T)\exp(-i\hat{H}_{\rm I}t)$ commutes with $\hat{F}_{\rm II}^A(T, t_0)$ even for $t \neq t_0$.

Equation (3.7) can be rewritten in the form

$$P(B_{\rm I}\in S, t; A_{\rm II}\in T, t_0|W_{\rm I}W_{\rm II}) = {\rm Tr}^{\rm I}\{\hat{F}_{\rm I}^B(S, t)\mathscr{F}_{\rm I}^A(T, t_0)\hat{W}_{\rm I}\} \tag{3.8}$$

where

$$\begin{aligned}\mathscr{F}(T, t_0)\hat{W}_{\rm I} &= {\rm Tr}^{\rm II}\{\hat{F}_{\rm II}^A(T, t_0)\hat{U}\hat{W}_{\rm I}\hat{W}_{\rm II}\hat{U}^*\}\\ &= {\rm Tr}^{\rm II}\{[\hat{F}_{\rm II}^A(T, t_0)]^{1/2}\hat{U}\hat{W}_{\rm II}^{1/2}\hat{W}_{\rm I}\hat{W}_{\rm II}^{1/2}\hat{U}^*[\hat{F}_{\rm II}^A(T, t_0)]^{1/2}\}\end{aligned} \tag{3.9}$$

defines an o.v.m. Note that

$$\hat{F}_{\rm I}^A(T, t_0) = [\mathscr{F}_{\rm I}^A(T, t_0)]'\hat{I} \tag{3.10}$$

Finally from equations (3.5) and (3.8) we obtain

$$P(B_{\rm I}\in S, t|A_{\rm I}\in T, t_0; W_{\rm I}) = \frac{\mathrm{Tr}^{\rm I}\{\hat{F}_{\rm I}^{B}(S, t)\mathscr{F}_{\rm I}^{A}(T, t_0)\hat{W}_{\rm I}\}}{\mathrm{Tr}^{\rm I}\{\mathscr{F}_{\rm I}^{A}(T, t_0)\hat{W}_{\rm I}\}} \tag{3.11}$$

for the probability of observing $B_{\rm I}\in S$ at t conditioned by having observed $A_{\rm I}\in T$ at t_0.

Equation (3.11) shows that the reduction of the state can be quite naturally understood as a consequence of the application of the other postulates of quantum mechanics to the compound system I + II and loses much of its striking peculiarities. Notice also that (3.6) and (3.9) provide explicit expressions for the e.v.m. and o.v.m. corresponding to a definite procedure in terms of the characteristics of the apparatus, once an appropriate physical meaning has been attached to the formal observable $A_{\rm II}$ corresponding to $\hat{F}_{\rm II}^{A}(T)$.

Since in the above treatment II is described like I purely in terms of ordinary quantum mechanics, it is clear that even any statement on the apparatus has a meaning only with reference to an observer. What the discussion shows is that the borderline between the object and the observer can be shifted arbitrarily toward the observer. Von Neumann obtained the same result at the price of postulating the relation $\hat{U}\varphi_r\Phi_0=\varphi_r\Phi_r$, where the φ_r's and Φ_r's are the eigenstates of $A_{\rm I}$ and $A_{\rm II}$ that are supposed to be set in one–one correspondence. But the assumption is clearly completely ad hoc and fictitious, in fact: (a) the interaction and so the operator $\hat{U}$ are not at our choice, (b) in the assumption only discrete spectrum quantities would be observable, and (c) even for discrete quantities significant examples can be produced for which the relation cannot be exactly satisfied. On the contrary, as we have seen, no artificial hypothesis is necessary if the concept of observable is generalized to the e.v.m.'s.

4. OBSERVATION OF THE COMPLETE HISTORY OF A SET OF QUANTITIES

As a consequence of equations (2.8) and (2.9), the *joint probability* of observing for A a *sequence of results* at certain subsequent times $t_0 < t_1 \cdots t_n$ can be written as

$$\begin{aligned} &P(A\in T_N, t_N; \ldots; A\in T_1, t_1; A\in T_0, t_0|W) \\ &\quad = \mathrm{Tr}[\mathscr{F}_{S_A}(T_N, t_N)\cdots\mathscr{F}_{S_A}(T_1, t_1)\mathscr{F}_{S_A}(T_0, t_0)\hat{W}] \end{aligned} \tag{4.1}$$

which generalizes a well-known formula by Wigner. Notice that, setting

$$\mathscr{F}(T_N, t_N; \ldots; T_0, t_0) = \mathscr{F}_{S_A}(T_N, t_N)\cdots\mathscr{F}_{S_A}(T_0, t_0) \tag{4.2}$$

and

$$\hat{F}(T_N, t_N; \ldots; T_0, t_0) = \mathscr{F}'(T_0, t_0) \cdots \mathscr{F}'_{S_A}(T_N, t_N)\hat{I} \tag{4.3}$$

we find that equation (4.1) takes the form

$$\begin{aligned} P(A \in T_N, t_N; \ldots; A \in T_0, t_0 | W) \\ = \mathrm{Tr}[\hat{F}(T_N, t_N; \ldots; T_0, t_0)\hat{W}] \\ = \mathrm{Tr}[\mathscr{F}(T_N, t_N; \ldots; T_0, t_0)\hat{W}] \end{aligned} \tag{4.4}$$

Since equations (4.4) obviously define an o.v.m. and e.v.m. on $\mathscr{B}(\mathbf{R}^{p(N+1)})$, a sequence of observations at subsequent times is treated on the same footing as a single observation at one time.

This last circumstance is particularly interesting, since it suggests the possibility of treating in a significant way the somehow limit situation of a system continuously kept under observation for a certain time interval. It is well known that such a limit situation would bring unavoidable paradoxes in the framework of ordinary textbook quantum mechanics.

In order to achieve this aim we have to generalize the concepts of e.v.m. and o.v.m. replacing the space $\mathbf{R}^p$ of the possible values for a set of quantities at a definite time by the functional space $\mathbf{Y}$ of the *possible complete histories* $x(t) \equiv (x^1(t), \ldots, x^n(t))$ for a similar set of quantities in an entire time interval (t_i, t_f). Correspondingly, we have also to replace the class $\mathscr{B}(\mathbf{R}^p)$ of subsets of $\mathbf{R}^p$ by an appropriate σ-algebra of subsets of $\mathbf{Y}$. For this purpose we find it convenient to set $t_i = -\infty$, $t_f = +\infty$ and to identify $\mathbf{Y}$ with the Cartesian product $\mathbf{E}' = \mathscr{D}' \times \cdots \times \mathscr{D}'$ of n identical factors $\mathscr{D}'$, $\mathscr{D}'$ being the space of the Schwarz ordinary distributions. Note that $\mathbf{E}'$ is the dual space of $\mathbf{E} = \mathscr{D} \times \cdots \times \mathscr{D}$, $\mathscr{D}$ being the space of the infinitely differentiable functions with compact support in $\mathbf{R}$.

For any given element $h(t) \equiv (h_1(t), \ldots, h_2(t))$ of $\mathbf{E}$ and any trajectory $x(t) \in \mathbf{E}'$ we may define the *time average*

$$x_h = \int dt\, h_s(t) x^s(t) \tag{4.5}$$

This quantity can also be assumed as a *coordinate* which partially specifies the trajectory. If we choose l different linearly independent elements of $\mathbf{E}$, $h^{(1)}, \ldots, h^{(l)}$, we may introduce l different coordinates $x_{h^{(1)}}, \ldots, x_{h^{(l)}}$ for $x(t)$ and correspondingly consider the subset of $\mathbf{E}'$,

$$C(h^{(1)}, \ldots, h^{(l)}; B_l) = \{x(t) \in \mathbf{E}' \colon (x_{h^{(1)}}, \ldots, x_{h^{(l)}}) \in B_l\} \tag{4.6}$$

B_l is a Borel set in $\mathbf{R}^l$. The subsets of the form (4.6) for any choice of l, B_l, and $h^{(1)}, \ldots, h^{(l)}$ are called *cylinder sets*. They generate a σ-algebra which we shall denote by Σ; furthermore, we shall denote by $\Sigma_{t_0}^{t_1}$ the σ-algebra generated by the cylinder sets for which $h^{(1)}, \ldots, h^{(l)}$ have support in the interval (t_0, t_1).

Then to the continuous observation of a set of quantities we associate a mathematical structure which we call a *operation-valued stochastic process* (OVSP) and denote by

$$\{\mathbf{E}', \Sigma_{t_0}^{t_1}, \mathscr{F}(t_1, t_0; \cdot)\}$$

Such a structure is defined in the following way.

1. For any time interval (t_0, t_1) an o.v.m. $\mathscr{F}(t_1, t_0; M)$ and a related e.v.m. $F(t_1, t_0; M) = \mathscr{F}'(t_1, t_0; M)\hat{I}$ are given on $\Sigma_{t_0}^{t_1}$ and the probability of observing a result $x(t) \in M$ is expressed by

$$P(M|W, t_0) = \mathrm{Tr}[F(t_1, t_0; M)\hat{W}] = \mathrm{Tr}[\mathscr{F}(t_1, t_0; M)\hat{W}] \tag{4.7}$$

if the system is prepared in the state W before the time t_0.

2. The composition law [cf. equation (4.2)]

$$\mathscr{F}(t_2, t_0; N \cap M) = \mathscr{F}(t_2, t_1; N)\mathscr{F}(t_1, t_0; M) \tag{4.8}$$

holds for $M \in \Sigma_{t_0}^{t_1}$ and $N \in \Sigma_{t_1}^{t_2}$ (note that $N \cap M \subset \Sigma_{t_0}^{t_2}$).

3. The conditional probability of finding $x(t) \in \Sigma_{t_1}^{t_2}$ if $x(t) \in M \in \Sigma_{t_0}^{t_1}$ has been observed is given by [cf. equation (3.11)]

$$P(N|M; W, t_0) = \mathrm{Tr}[\hat{F}(t_2, t_1; N)\mathscr{F}(t_1, t_0; M)\hat{W}]/\mathrm{Tr}[\mathscr{F}(t_1, t_0; M)\hat{W}] \tag{4.9}$$

4. The time translation equation [cf. equation (2.7b)]

$$\mathscr{F}(t_1 + \tau, t_0 + \tau; M_\tau)\hat{X} = e^{i\hat{H}\tau}[\mathscr{F}(t_1, t_0; M)(e^{-i\hat{H}\tau}\hat{X}e^{i\hat{H}\tau})]e^{-i\hat{H}\tau} \tag{4.10}$$

holds, where $M_\tau = \{x(t); x(t) = x'(t - \tau), x'(t) \in M\}$.

5. $\mathscr{F}(t_1, t_0; M)$ is normalized; i.e., if we set $\mathscr{G}(t_1, t_0) = \mathscr{F}(t_1, t_0; \mathbf{E}')$, the equation

$$\mathrm{Tr}[\mathscr{G}(t_1, t_0)\hat{X}] = \mathrm{Tr}\,\hat{X} \tag{4.11}$$

holds.

Note that if we put $M = \mathbf{E}'$ in equation (4.9), by equation (4.11) we have

$$P(N|\mathbf{E}'; W, t_0) = \mathrm{Tr}[\hat{F}(t_2, t_1; N)\mathscr{G}(t_1, t_0)\hat{W}] \tag{4.12}$$

So the mapping $\mathscr{G}(t_1, t_0)$ describes the modification produced on the state of the system by the action of the apparatus when no notice is taken of the result; briefly it describes the *disturbance* by the apparatus.

5. CHARACTERISTIC FUNCTIONAL, POISSONIAN AND GAUSSIAN OVSP

In the preceding section we have introduced a formalism for treating continuous observations on an axiomatic basis. It remains to show that an object $\mathscr{F}(t_1, t_0; M)$ satisfying all the requirements we have introduced actually exists and to produce significant examples.

For this purpose we find it convenient to introduce the *characteristic functional* related to the probability distribution defined by equation (4.7). We set

$$L(t_1, t_0; [\xi(t)]\,|W) = \int dP([x(t)]\,|W, t_0) \exp\left\{i \int_{t_0}^{t_1} dt\, \xi_s(t) x^s(t)\right\} \quad (5.1)$$

for any $\xi(t) \in \mathbf{E}$. Such a quantity has the following important properties:

1. *Positivity* [it follows by the positivity of $P(M|Wt_0)$],

$$\sum_{ij} c_i^* L(t_1, t_0; [\xi^{(i)}(t) - \xi^{(j)}(t)]\,|W) c_j \geq 0 \quad (5.2)$$

for any choice of the test functions $\xi^{(1)}(t), \xi^{(2)}(t), \ldots$ and of the complex numbers $c_1, c_2, \ldots$.

2. *Normalization* [it follows by (4.11)]

$$L(t_1, t_0; 0|W) = 1 \quad (5.3)$$

A general theorem (Milnos' Theorem) in the theory of the so-called *generalized stochastic process* states that conversely if a functional $L(t_1, t_0; [\xi(t)]\,|W)$ satisfies (5.2), (5.3), and certain regularity conditions, it is the characteristic functional of a probability distribution $P(M|W; t_0)$. In practice, first we construct the probability density for the quantities $x_{h^{(1)}}, \ldots, x_{h^{(l)}}$ as

$$\begin{aligned} &p(x_1, h^{(1)}; \ldots; x_l, h^{(l)}|Wt_0) \\ &= \frac{1}{(2\pi)^l} \int dk_1 \cdots dk_l \exp\left(-i \sum_{j=1}^{l} k_j x_j\right) L\left(t_1, t_0; \left[\sum_{j=1}^{l} k_j h^{(j)}(t)\right]\right) \\ &\equiv \int dP([x(t)]\,|W, t_0)\delta(x_1 - x_{h^{(1)}}) \cdots \delta(x_l - x_{h^{(l)}}) \end{aligned} \quad (5.4)$$

Then we obtain the probability for a cylinder set $P(C(h^{(1)} \cdots h^{(l)}; B_l)\,|Wt_0)$ and finally we extend it to the entire $\Sigma_{t_0}^{t_1}$.

Mimicking the above procedure, we may define a *characteristic functional operator* (CFO)

$$\mathscr{G}(t_1, t_0; [\xi(t)]) = \int d\mathscr{F}(t_1, t_0; [x(t)]) \exp\left\{i \int_{t_0}^{t_1} dt\, \xi_s(t) x^s(t)\right\} \quad (5.5)$$

which is related to L by the equation

$$L(t_1, t_0; [\xi(t)] \,|W) = \mathrm{Tr}\{\mathscr{G}(t_1, t_0; [\xi(t)])\hat{W}\} \tag{5.6}$$

and which has the following properties:

$$\sum_{ij} c_i^* \mathscr{G}(t_1, t_0; [\xi^{(i)}(t) - \xi^{(j)}(t)])c_j: \quad \text{completely positive} \tag{5.7}$$

$$\mathscr{G}(t_1, t_0; \mathbf{0}) = \mathscr{F}(t_1, t_0; \mathbf{E}') = \mathscr{G}(t_1, t_0): \quad \text{trace preserving} \tag{5.8}$$

which correspond to (5.2) and (5.3).

Furthermore, if $\xi_1(t)$ and $\xi_2(t)$ are two elements of $\mathbf{E}$ with support in (t_0, t_1) and (t_1, t_2), respectively, from equation (4.8) we have

$$\mathscr{G}(t_2, t_0; [\xi_1(t) + \xi_2(t)]) = \mathscr{G}(t_2, t_1; [\xi_2(t)])\mathscr{G}(t_1, t_0; [\xi_1(t)]) \tag{5.9}$$

while a time translation equation similar to (4.10) can be written also for $\mathscr{G}$.

Then, if we have a mapping $\mathscr{G}(t_1, t_0; [\xi(t)])$ in $\mathbf{T(H)}$ satisfying (5.7)–(5.9) and the time translation equation, an associated $\mathscr{F}(t_1, t_0; M)$ with all the required properties can be constructed starting from the operatorial equation corresponding to equation (5.4). The problem of constructing on OVSP is thus reduced to the simpler problem of constructing a *characteristic functional operator*.

In order to solve the last problem let us introduce two additional hypotheses (which obviously amount to restricting the class of CFO we are able to take into consideration). First we assume that $\mathscr{G}(t_1, t_0; [\xi])$ can be extended to functions not vanishing at t_0 and t_1; then, for any $\xi(t) \in \mathbf{E}$, equation (5.9) can be written as

$$\mathscr{G}(t_2, t_0; [\xi(t)]) = \mathscr{G}(t_2, t_1; [\xi(t)])\mathscr{G}(t_1, t_0; [\xi(t)]) \tag{5.10}$$

Furthermore, we assume that (5.10) can be put in the differential form

$$\frac{\partial}{\partial t}\mathscr{G}(t, t_0; [\xi(\tau)]) = \mathscr{K}(t; \xi(t))\mathscr{G}(t, t_0; [\xi(\tau)]) \tag{5.11}$$

from which $\mathscr{G}(t_1, t_0; [\xi(t)])$ can be reobtained as

$$\mathscr{G}(t_1, t_0; [\xi(t)]) = T \exp \int_{t_0}^{t_1} dt\, \mathscr{K}(t; \xi(t)) \tag{5.12}$$

T denoting the time ordering prescription.

The problem is now to characterize the class of the operators $\mathscr{K}(t; \xi(t))$ for which $\mathscr{G}(t_1, t_0; [\xi(t)])$ as given by (5.12) satisfies (5.7) and (5.8). We are not able to solve this problem in full generality, but we can produce an already interesting subclass for which the above conditions are met.

First let us set $\xi(t) = 0$ in (5.11). Then we obtain

$$\frac{\partial}{\partial t}\mathscr{G}(t, t_0) = \mathscr{L}(t)\mathscr{G}(t, t_0) \tag{5.13}$$

with $\mathscr{L}(t) = \mathscr{K}(t; 0)$ and we must find under what assumptions on $\mathscr{L}(t)$ the mapping $\mathscr{G}(t, t_0)$ defined by (5.13) turns out to be completely positive and trace preserving [cf. (5.8)]. This last problem has been studied in a different context (Gorini *et al.*, 1976; Lindblad, 1976). It is found that, if $\mathscr{L}(t)$ is bounded, it must be of the form

$$\mathscr{L}(t)\hat{X} = -i[\hat{K}(t), \hat{X}] - \frac{1}{2}\sum_{j=1}^{Q} [\hat{R}_j^*(t)\hat{R}_j(t), \hat{X}]_+ + \sum_{j=1}^{Q} \hat{R}_j(t)\hat{X}\hat{R}_j^*(t) \tag{5.14}$$

with $\hat{K}(t) = \hat{K}^*(t)$ and $\hat{R}_1(t), \hat{R}_2(t), \ldots$ bounded operators. If $\mathscr{L}(t)$ is not bounded, equation (5.14) (with $\hat{K}, \hat{R}_1, \ldots$ not bounded) turns out to be still a sufficient condition in order that $\mathscr{G}(t, t_0)$ has the two required properties (apart from some pathological cases). Once (5.14) has been assumed, it can be shown that even (5.7) is satisfied if in turn $\mathscr{K}(t, \xi(t))$ is assumed to be of the form

$$\begin{aligned}\mathscr{K}(t, \xi(t))\hat{X} = \mathscr{L}(t)\hat{X} &+ \sum_{j=1}^{P} (e^{i\alpha_j^s \xi_s(t)} - 1)\hat{R}_j(t)\hat{X}\hat{R}_j^*(t) \\ &+ \sum_{j=P+1}^{Q} \{i\alpha_j^s\xi_s(t)[\hat{R}_j(t)\hat{X} + \hat{X}\hat{R}_j^*(t)] \\ &- \tfrac{1}{2}[\alpha_j^s\xi_s(t)]^2\hat{X}\} + i\beta^s\xi_s(t)\hat{X}\end{aligned} \tag{5.15}$$

(where $\alpha_1, \ldots, \alpha_\mu, \beta$ are arbitrary vectors in $\mathbf{R}^n$). This last result has been obtained in the above generality using techniques developed in the so-called quantum stochastic calculus (Barchielli and Lupieri, 1985, 1986).

For the sake of analogy with the numerical stochastic process and for reasons which shall be apparent in a moment, the second term in (5.15) is said to be the *Poissonian term*, while the third one is said to be the *Gaussian term*.

Notice finally that $\hat{K}(t), \hat{R}_1(t), \ldots$ must be simply Heisenberg operators $\hat{K}(t) = \exp(i\hat{H}t)\,\hat{K}\exp(-i\hat{H}t)$, etc., in order for the time translation prescription to be satisfied. Furthermore, $\hat{K}$ itself can be often reabsorbed in a redefinition of $\hat{H}$ and without loss of generality can be assumed to vanish.

Let us now try to understand the meaning of the result we have obtained. Notice that from (5.1) it follows that

$$\langle x^{s_1}(t^{(1)})x^{s_2}(t^{(2)}) \cdots x^{s_l}(t^{(l)})\rangle = (-i)^l \frac{\delta^l L(t_1, t_0; [\xi(t)]\,|W)}{\delta\xi_{s_1}(t^{(1)}) \cdots \delta\xi_{s_l}(t^{(l)})}\bigg|_{\xi=0} \tag{5.16}$$

Then, combining such an equation with (5.6) and (5.12), we obtain in particular

$$\langle x^s(t)\rangle = -i\,\mathrm{Tr}\left\{\frac{\partial\mathscr{K}(t,\xi)}{\partial\xi_s}\mathscr{G}(t,t_0)\hat{W}\right\}\bigg|_{\xi=0} = \mathrm{Tr}\{\hat{O}^s(t)\mathscr{G}(t,t_0)\hat{W}\} \tag{5.17}$$

with

$$\hat{O}^s = \sum_{j=1}^{P} \alpha_j^s \hat{R}_j^* \hat{R}_j + \sum_{j=P+1}^{Q} \alpha_j^s(\hat{R}_j + \hat{R}_j^*) + \beta^s \tag{5.18}$$

Equation (5.18) is analogous to equation (2.10) and qualifies our OVSP as corresponding to a (coarse-grained) continuous observation of the set of quantities related to the operators $\hat{O}^1, \hat{O}^2, \ldots, \hat{O}^l$. Notice, however, that we can write similarly

$$\begin{aligned}
&\langle x^s(t)x^{s'}(t')\rangle \\
&= \delta(t-t')\,\mathrm{Tr}\left\{\frac{\delta^2\mathscr{K}(t,\xi)}{\delta\xi_s\delta\xi_{s'}}\mathscr{G}(t,t_0)\hat{W}\right\}\bigg|_{\xi=0} \\
&\quad + \theta(t-t')\,\mathrm{Tr}\left\{\frac{\delta\mathscr{K}(t,\xi)}{\delta\xi^s}\mathscr{G}(t,t')\frac{\delta\mathscr{K}(t',\xi)}{\delta\xi^{s'}}\mathscr{G}(t',t_0)\hat{W}\right\}\bigg|_{\xi=0} \\
&\quad + \theta(t'-t)\,\mathrm{Tr}\left\{\frac{\delta\mathscr{K}(t',\xi)}{\delta\xi^{s'}}\mathscr{G}(t',t)\frac{\delta\mathscr{K}(t,\xi)}{\delta\xi^s}\mathscr{G}(t,t_0)\hat{W}\right\}\bigg|_{\xi=0}
\end{aligned} \tag{5.19}$$

and the occurrence of the $\delta(t-t')$ in the first term of such an equation shows that only time averages of the kind defined by (4.5) have an actual meaning and that for nonsufficiently smooth weight functions the expected fluctuations are very large.

Notice also that in the pure Poissonian case $P = Q$ the operators $\hat{O}^s$, as defined by (5.18), are positive definite, while in the Gaussian case they can have *a priori* eigenvalues with both signs.

6. A SIMPLE MODEL FOR THE CHOICE OF THE MACROSCOPIC VARIABLES

Let us try to apply the above formalism to the macroscopic description of a large body. For simplicity let us consider a system of identical particles with a strictly local interaction.

We can use the formalism of the second quantization, characterized by the Hamiltonian

$$\hat{H} = \int d^3\mathbf{x}\left[\frac{1}{2m}\partial_k\psi^*(\mathbf{x}, t)\partial_k\psi(\mathbf{x}, t) + \frac{\lambda}{2}\psi^*(\mathbf{x}, t)\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)\psi(\mathbf{x}, t)\right] \tag{6.1}$$

and the commutation relations

$$\begin{aligned}&[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)]_\pm = [\psi^*(\mathbf{x}, t), \psi^*(\mathbf{x}', t)]_\pm = 0\\ &[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t)]_\pm = \delta^3(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{6.2}$$

The simplest quantity that can be used for a specification of the macroscopic status of the system is the density of particles $n(\mathbf{x}, t)$. We shall associate the macroscopic density to the purely Poissonian OVSP defined by the CFO [cf. (5.15)]

$$\begin{aligned}\mathscr{K}(t, \xi) = \gamma \int d^3\mathbf{x}\, \{&e^{(i/\gamma)\xi(\mathbf{x})}\psi(\mathbf{x}, t)\cdot\psi^*(\mathbf{x}, t)\\ &-\tfrac{1}{2}[\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t), \cdot\,]_+\}\end{aligned} \tag{6.3}$$

In fact we have

$$\hat{n}(\mathbf{x}, t) = -i\left\{\frac{\delta\mathscr{K}(t, \xi)}{\delta\xi(\mathbf{x}, t)}\right\}'_{\xi=0}\hat{I} = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t) \tag{6.4}$$

and

$$\langle n(\mathbf{x}, t)\rangle = \mathrm{Tr}\{\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)\mathscr{G}(t, t_0)\hat{W}\} \tag{6.5}$$

Then we can study the conservation properties of n for the time evolution statistically implied by (6.3). Using the usual operatorial equation

$$\frac{\partial\hat{n}}{\partial t} + \partial_k\hat{J}_k = 0 \tag{6.6}$$

where

$$\hat{J}_k(\mathbf{x}, t) = -\frac{i}{2m}(\psi^*\partial_k\psi - \partial_k\psi^*\psi) \tag{6.7}$$

we have

$$\begin{aligned}\frac{\partial\langle n(\mathbf{x}, t)\rangle}{\partial t} &= \mathrm{Tr}\left\{\frac{\partial\hat{n}(\mathbf{x}, t)}{\partial t}\mathscr{G}(t, t_0)\hat{W} + \hat{n}(\mathbf{x}, t)\frac{\partial\mathscr{G}(t, t_0)}{\partial t}\hat{W}\right\}\\ &= -\partial_k\langle J_k(\mathbf{x}, t)\rangle_{\mathrm{QM}} + \mathrm{Tr}\{\hat{n}(\mathbf{x}, t)\mathscr{L}(t)\mathscr{G}(t, t_0)\hat{W}\}\end{aligned} \tag{6.8}$$

Finally, taking into account that

$$\begin{aligned}\mathscr{L}'\hat{n}(\mathbf{x}, t) &= \gamma \int d^3\mathbf{x}'\{\psi^*(\mathbf{x}', t)\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)\psi(\mathbf{x}', t) \\ &\quad -\tfrac{1}{2}[\psi^*(\mathbf{x}', t)\psi(\mathbf{x}', t), \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)]_+\} \\ &= -\gamma\hat{n}(\mathbf{x}, t)\end{aligned} \tag{6.9}$$

we obtain

$$\frac{\partial\langle n\rangle}{\partial t} + \partial_k\langle J_k\rangle_{\text{QM}} = -\gamma\langle n\rangle \tag{6.10}$$

having set

$$\langle J_k\rangle_{\text{QM}} = \text{Tr}\{\hat{J}_k\mathscr{G}(t, t_0)\hat{W}\} \tag{6.11}$$

Notice that equation (6.11) provides the most direct analog of the ordinary quantum mechanical expectation value (obviously $\langle n\rangle = \langle n\rangle_{\text{QM}}$).

Equation (6.10) shows that the expectation value of the number of particles is not conserved in the present formalism and for the chosen OVSP; the decay time is $\tau = 1/\gamma$.

Formally similar equations can be written for the energy density

$$\hat{w} = \frac{1}{2m}\left(\partial_k\psi^*\partial_k\psi + \frac{\lambda}{2}\psi^*\psi^*\psi\psi\right) \tag{6.12}$$

and for the momentum density

$$\hat{g}_k = -\frac{i}{2}(\psi^*\partial_k\psi - \partial_k\psi^*\psi) \tag{6.13}$$

We have

$$\begin{aligned}\partial_t\langle g_k\rangle_{\text{QM}} + \partial_h\langle T_{hk}\rangle_{\text{QM}} &= -\gamma\langle g_k\rangle_{\text{QM}} \\ \partial_t\langle w\rangle_{\text{QM}} + \partial_h\langle S_h\rangle_{\text{QM}} &= -\gamma\langle w\rangle_{\text{QM}}\end{aligned} \tag{6.14}$$

where T_{hk} and S_k denote the usual momentum tensor and energy current, and the expectation values are defined directly as in (6.11) without introducing new quantities in the form of the OVSP. In this sense even the energy and the momentum are not conserved, the decay time being the same as for the number of particles.

Let us try to understand how important the above dissipative terms might be, that is, how large τ can be assumed.

For $t' \sim t$ the analog of (5.19) can be written

$$\langle n(\mathbf{x}, t)n(\mathbf{x}', t')\rangle = \frac{1}{\gamma}\delta(t - t')\delta^3(\mathbf{x} - \mathbf{x}')\langle n(\mathbf{x}, t)\rangle + \langle n(\mathbf{x}, t)n(\mathbf{x}', t')\rangle_{\text{QM}} \tag{6.15}$$

Denoting by $\bar{n}$ an average of $n(\mathbf{x}, t)$ over a small volume ΔV and a small interval of time Δt, we obtain from the above equation

$$\langle \bar{n}^2 \rangle = \frac{1}{\gamma} \frac{1}{\Delta V \Delta t} \langle \bar{n} \rangle + \langle \bar{n}^2 \rangle_{QM} \tag{6.16}$$

In order that the macroscopic fluctuation remain small it is necessary that the first term in (6.16) is small with respect to $\bar{n}^2$ for typical values of $\bar{n}$; that is,

$$\tau \equiv \frac{1}{\gamma} \ll \bar{n} \Delta V \Delta t \tag{6.17}$$

Since for a gas in ordinary conditions we have $n \sim 3 \times 10^{19}\ \mathrm{cm}^{-3}$, it is clear that τ must be small with respect to 10^{19} sec (to be compared with the life time of the universe, of the order of 10^{17} sec).

The above value seems to be too small to take the model seriously at a cosmological time scale. On ordinary time scales, however, we can assume consistently that the right-hand sides of (6.10) and (6.14) are negligible and we can use such equations as a starting point for the construction of thermodynamics. According to a usual strategy in nonequilibrium statistical mechanics, we can define a field of velocity $v_k(\mathbf{x}, t)$, a pressure tensor P_{hk}, a density of internal energy u, and a heat current q_k by

$$\begin{aligned} nmv_k &= m\langle J_k \rangle_{QM} = \langle g_k \rangle_{QM} \\ \langle T_{hk} \rangle_{QM} &= nmv_h v_k + P_{hk}, \qquad \langle w \rangle_{QM} = \tfrac{1}{2} nmv^2 \\ \langle S_k \rangle_{QM} &= (\tfrac{1}{2} nmv^2 + u) v_k + P_{kh} v_h + q_k \end{aligned} \tag{6.18}$$

and then, using the thermal state equation $u = u(n, T)$, try to reexpress P_{hk} in terms of n, v_k, and the temperature T by means of appropriate approximations (including the possibility of neglecting the fluctuations of the considered quantities).

The interesting aspect with respect to the usual formalism is that now the basic macroscopic quantity $n(\mathbf{x}, t)$, from which all others have been derived, can be thought of as having well-defined values for any $\mathbf{x}$ and any t, even if the theory makes only statistical predictions.

The model can be immediately extended to the case of various types of particles and so made more realistic. Even the assumption of a strictly local interaction is obviously not essential, if the various densities introduced are replaced by appropriate space smearings. On the contrary, a direct relativistic generalization does not seem possible. For instance, in the case of fermions, the relativistic scalar most analogous to n would be the quantity $\bar{\psi}\psi$, which can be interpreted as the difference between the density of particles and of antiparticles in the CM. However, $\bar{\psi}\psi$ is not positive definite and cannot correspond to a Poissonian OVSP; indeed, if we try to generalize (6.1) replacing ψ^* with $\bar{\psi}$, we would not obtain a positive generator.

7. CONCLUDING REMARKS

In conclusion, the effect-operation formalism satisfies exactly the von Neumann consistency requirement on the shift of the borderline between object and apparatus. However, it does not allow one to eliminate such a borderline and can be applied only to intervals of time that have to be fixed *a priori* in some way. An important consequence of the above circumstances is that quantum mechanics cannot be applied consistently to the entire universe even in this more sophisticated formulation.

On the contrary, the model considered in Section 6 shows that in the formalism of the OVSP it is possible to introduce some basic macroscopic quantities as "be-ables," that is, as having well-defined values at any place and at any time. Then one can identify the system II of Section 3 with an appropriate large body described in terms, e.g., of the densities $n_1, n_2, \ldots$ of its various components (e.g., electrons and various types of nuclei) and replace the o.v.m. $\mathscr{F}_{\mathrm{II}}^A$ by the OVSP of the model, the interaction with I resulting in a modification of the values of $n_1, n_2, \ldots$. Obviously, intermediate microscopic systems between I and II can be considered, but, once the level of the macroscopic description is reached in the von Neumann shift, the explicit reference to the observer is no longer necessary. We can talk in terms of objective events and in particular the application of the theory to the entire universe no longer raises difficulties. Naturally in this view the choice of the basic quantities relative to the large system cannot be arbitrary, but it should be prescribed by an appropriate postulate to be considered a part of the theory and γ should be a new fundamental constant.

As we have seen, the price one has to pay for the above result is a violation of the conservation laws, which does not seem acceptable on a large scale of time. Furthermore, an immediate relativistic generalization does not seem possible [for an example of a relativistic extension of the OVSP formalism in the Gaussian case see Barchielli *et al.* (1984)]. In spite of this I think the model can be interesting for the illustration of a possible philosophy in approaching the problem of measurement in quantum mechanics.

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